Dimension bounds on sutured instanton homology

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Sketch

For a balanced sutured manifold (M, γ), one can define

- sutured (Heegaard) Floer homology $SFH(M, \gamma)$ by Juhász '06, a $\mathbb{Z}$-module or an $\mathbb{F}_2$-vector space, whose chain complex is $SFC(M, \gamma)$;
- sutured monopole homology $SHM(M, \gamma)$ by Kronheimer-Mrowka '10, a $\mathbb{Z}$-module or a module of the Novikov ring over $\mathbb{Z}_2$;
- sutured instanton homology $SHI(M, \gamma)$ by Kronheimer-Mrowka '10, a $\mathbb{C}$-vector space.

We prove

- $\dim_{\mathbb{C}} SHI(M, \gamma) \leqslant \dim_{\mathbb{F}_2} SFC(M, \gamma)$ (Baldwin-Li-Y. '20);
- $\chi_{\mathrm{gr}}(SHI(M, \gamma)) = \chi_{\mathrm{gr}}(SFH(M, \gamma))$ (Li-Y. '21);
- similar results for $SHM(M, \gamma)$.

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Construction of SFH

Definition (Gabai '83, Juhász '06)

A **balanced sutured manifold** (M, γ) consists of a compact oriented 3-manifold M with non-empty boundary together with an oriented closed 1-submanifold γ on ∂M, which satisfies the following conditions.

- $R(\gamma) = \partial M \backslash \mathrm{int}(N(\gamma))$. Neither M nor $R(\gamma)$ has a closed component.
- $R(\gamma) = R_+(\gamma) \sqcup R_-(\gamma)$, where the orientations induced by ∂M and γ are the same on $R_+(\gamma)$, but different on $R_-(\gamma)$.
- (Balanced condition) $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

Examples of balanced sutured manifolds

- (B^3, δ), where $\delta \subset \partial B^3$ is an oriented circle.
- $(Y - \mathrm{int}B^3, \delta)$, where Y is a connected closed 3-manifold.
- $(Y - \mathrm{int}N(K), m \cup (-m))$, where K is a knot and m is a meridian of K.

Construction of SFH

Definition (Juhász '06)

A **balanced diagram** $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a tuple satisfying the following.

- Σ is a compact, oriented surface with boundary.
- $\alpha = \{\alpha_1, \dots, \alpha_n\}$ and $\beta = \{\beta_1, \dots, \beta_n\}$ are two sets of pairwise disjoint simple closed curves in the interior of Σ.
- The maps $\pi_0(\partial\Sigma) \to \pi_0(\Sigma \backslash \alpha)$ and $\pi_0(\partial\Sigma) \to \pi_0(\Sigma \backslash \beta)$ are surjective.

From a balanced diagram, we can construct a sutured manifold as follows.

Construction of SFH

Suppose $\mathcal{H} = (\Sigma, \alpha, \beta)$ is an (admissible) balanced diagram with $n = |\alpha| = |\beta|$. Consider two tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_n \text{ and } \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_n$$

in the symmetric product

$$\mathrm{Sym}^n\Sigma = (\prod_{i=1}^{n} \Sigma)/S_n, \text{ where } S_n \text{ is the symmetric group.}$$

The chain complex $SFC(\mathcal{H})$ is a $\mathbb{Z}$-module generated by intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, *i.e.*

$$\boldsymbol{x} = x_1 \times \cdots \times x_n \text{ for } x_i \in \alpha_i \cap \beta_{\sigma(i)}, \sigma \in S_n.$$

Construction of SFH

Roughly speaking, $\mathrm{Sym}^n\Sigma$ is a symplectic manifold and $\mathbb{T}_\alpha, \mathbb{T}_\beta$ are Lagrangian submanifolds. Let $SFH(\mathcal{H})$ be the Lagrangian Floer homology of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

Theorem (Juhász '06)

For a balanced sutured manifold (M, γ), different choices of the balanced diagram $\mathcal{H}$ induce isomorphic $\mathbb{Z}$-modules $SFH(\mathcal{H})$, denoted by $SFH(M, \gamma)$.

Remark

Juhász-Thurston-Zemke '12 proved the naturality of $SFH(M, \gamma)$ over $\mathbb{F}_2$.

Remark

$$SFH(Y - \mathrm{int}B^3, \delta) \cong \widehat{HF}(Y),$$
$$SFH(Y - \mathrm{int}N(K), m \cup (-m)) \cong \widehat{HFK}(Y, K).$$

Construction of SFH

$SFH(M,\gamma)$ admits a spin^c decomposition

$$SFH(M,\gamma)=\bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)} SFH(M,\gamma,\mathfrak{s}),$$

where $\mathrm{Spin}^c(M,\gamma)$ is an affine space over $H^2(M,\partial M;\mathbb{Z})\cong H_1(M;\mathbb{Z})=H'$.

There is also a relative $\mathbb{Z}_2$-grading on $SFH(M,\gamma,\mathfrak{s})$. Fix $\mathfrak{s}_0\in\mathrm{Spin}^c(M,\gamma)$.

$$\chi(SFH(M,\gamma))=\sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)\\ \mathfrak{s}-\mathfrak{s}_0=h\in H^2(M,\partial M;\mathbb{Z})}} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot\mathrm{PD}(h)\in\mathbb{Z}[H']/\pm H',$$

where $\mathrm{PD}:H^2(M,\partial M;\mathbb{Z})\to H_1(M;\mathbb{Z})$ is the Poincaré duality map.

Construction of SFH

Theorem (Friedl-Juhász-Rasmussen '11)

Suppose $H' = H_1(M;\mathbb{Z})$. We have

$$\chi(SFH(M,\gamma)) = \tau(M,\gamma) \in \mathbb{Z}[H']/\pm H',$$

where $\tau(M,\gamma)$ is a Turaev-type torsion and can be calculated by Fox calculus.

Remark

Later, we will consider

$$\chi_{\mathrm{gr}}(SFH(M,\gamma)) = p_*(\chi(SFH(M,\gamma))) \in \mathbb{Z}[H]/\pm H,$$

where p_* is induced by $p : H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})/\mathrm{Tors} = H$.

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Constructions of SHM and SHI

Suppose Y is a closed 3-manifold. Based on Seiberg-Witten equations, Kronheimer-Mrowka '07 constructed three versions of monopole Floer homology

$$\widehat{HM}_\bullet(Y) \text{ (from)}, \ \widecheck{HM}_\bullet(Y) \text{ (to)}, \ \overline{HM}_\bullet(Y) \text{ (bar)}.$$

They admit a spin^c decomposition over $\mathrm{Spin}^c(Y)$. For any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ with $c_1(\mathfrak{s})$ nontorsion, Kronheimer-Mrowka showed that

$$\overline{HM}_\bullet(Y,\mathfrak{s}) = 0 \ \text{ and } \ \widehat{HM}_\bullet(Y,\mathfrak{s}) \cong \widecheck{HM}_\bullet(Y,\mathfrak{s}).$$

For a closed surface $R \subset Y$ with $g(R) \geqslant 2$, define

$$HM(Y|R) = \bigoplus_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(Y) \\ \langle c_1(\mathfrak{s}), R\rangle = 2g(R)-2}} \widecheck{HM}_\bullet(Y,\mathfrak{s})$$

Constructions of SHM and SHI

Suppose Y is a closed 3-manifold and $\omega \to Y$ is a Hermitian line bundle such that $c_1(\omega)$ has odd pairing with some integer homology class. Based on Yang-Mills equations (related to $SO(3)$ connections), Floer '88 constructed a well-defined instanton homology group $I^\omega(Y)$.

For any homology class h in $H_*(Y)$, there is an action $\mu(h)$ on $I^\omega(Y)$. In particular, for a point pt and a closed surface R in Y, there are actions $\mu(\mathrm{pt})$ and $\mu(R)$, which commute with each other. Define

$$I^\omega(Y|R)$$

to be the simultaneous (generalized) eigenspace of $(\mu(\mathrm{pt}), \mu(R))$ with eigenvalues

$$(2, 2g(R) - 2).$$

Constructions of SHM and SHI

Definition (Kronheimer-Mrowka '10)

Suppose (M,γ) is a balanced sutured manifold. Let T be a connected compact surface with $\#\partial T=\#\gamma$. Let the **preclosure** $\widetilde{M}$ of (M,γ) be

$$\widetilde{M}=M\underset{\gamma=-\partial T}{\cup}[-1,1]\times T.$$

The boundary of $\widetilde{M}$ consists of two components

$$\widetilde{R}_+=R_+(\gamma)\cup\{1\}\times T\ \text{ and }\ \widetilde{R}_-=R_-(\gamma)\cup\{-1\}\times T.$$

Let $h:\widetilde{R}_+\cong\widetilde{R}_-$ be a diffeomorphism which reverses the boundary orientations. Let Y be the 3-manifold obtained from $\widetilde{M}$ by gluing $\widetilde{R}_+$ to $\widetilde{R}_-$ by h and let R be the image of $\widetilde{R}_+$ and $\widetilde{R}_-$ in Y. The pair (Y,R) is called a **closure** of (M,γ). The genus of R is called the **genus** of the closure (Y,R).

Constructions of SHM and SHI

A closure (Y, R) of (M, γ).

Definition (Kronheimer-Mrowka '10)

Suppose (Y, R) is a closure of (M, γ) with $g(R) \geqslant 2$. Define

$$SHM(M, \gamma) = HM(Y|R).$$

Definition (Kronheimer-Mrowka '10)

In the construction of the closure, suppose p is a point on T and the diffeomorphism h sends $\{1\} \times p$ to $\{-1\} \times p$. Let ω be the Hermitian line bundle such that $c_1(\omega)$ is Poincaré dual to $[-1, 1] \times p/ \sim_h$. For such a closure (Y, R, ω) with $g(R) \geqslant 1$, define

$$SHI(M, \gamma) = I^{\omega}(Y|R).$$

Constructions of SHM and SHI

Theorem (Kronheimer-Mrowka '10)

The isomorphism classes of $SHM(M,\gamma)$ and $SHI(M,\gamma)$ are independent of the choices of the surface T, the diffeomorphism h, and the point p.

Remark

Baldwin-Sivek '15 proved the naturality of $SHM(M,\gamma)$ and $SHI(M,\gamma)$.

Remark

$$SHM(Y-\mathrm{int}B^3,\delta)\cong \widetilde{HM}(Y) \text{ and } SHI(Y-\mathrm{int}B^3,\delta)\cong I^\sharp(Y).$$

Define

$$KHM(Y,K)=SHM(Y-\mathrm{int}N(K),m\cup(-m)),$$
$$KHI(Y,K)=SHI(Y-\mathrm{int}N(K),m\cup(-m)).$$

Constructions of SHM and SHI

For a basis $S_1, \dots, S_n$ of $H_2(M, \partial M)$, Li '19, Ghosh-Li '19 constructed a $\mathbb{Z}^n$-grading on $SHM(M,\gamma)$ and $SHI(M,\gamma)$. Suppose

$$\rho_1, \dots, \rho_n \in H^2(H, \partial M)/\mathrm{Tors}$$

are the dual basis of $S_1, \dots, S_n$. Define $\chi_{\mathrm{gr}}(SHM(M,\gamma))$ to be

$$\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} \chi(SHM(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n))) \cdot (\rho_1^{i_1} \cdots \rho_n^{i_n}).$$

It is a well-defined element in $\mathbb{Z}[H]/\pm H$, where

$$H = H_1(M)/\mathrm{Tors} \cong H^2(M, \partial M)/\mathrm{Tors}.$$

Define $\chi_{\mathrm{gr}}(SHI(M,\gamma))$ similarly.

Main theorem

Theorem A (Baldwin-Li-Y. '20)

Suppose (M, γ) is a balanced sutured manifold and $\mathcal{H}$ is an (admissible) balanced diagram of (M, γ). Then we have

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leqslant \dim_{\mathbb{F}_2} SFC(\mathcal{H}).$$

Theorem B (Li-Y. '21)

Suppose (M, γ) is a balanced sutured manifold and $H = H_1(M; \mathbb{Z})/\mathrm{Tors}$. Then

$$\chi_{\mathrm{gr}}(SHI(M, \gamma)) = \chi_{\mathrm{gr}}(SFH(M, \gamma)) \in \mathbb{Z}[H]/\pm H.$$

Remark

- Li-Y. '20 proved Theorem A for $(1, 1)$ diagrams of knots in lens spaces.
- In preparation (Li-Y.): Theorem B can be proved for $H' = H_1(M; \mathbb{Z})$.

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Motivation

Conjecture (Kronheimer-Mrowka '10)

Suppose (M,γ) is a balanced sutured manifold and Λ is the Novikov ring over $\mathbb{Z}_2$.

$$SHM(M,\gamma)\cong SFH(M,\gamma)\otimes\Lambda, SHI(M,\gamma)\cong SFH(M,\gamma)\otimes\mathbb{C}.$$

Theorem (Lekili '13, Baldwin-Sivek '16)

The above conjecture holds for SHM.

Remark

The proof is based on the isomorphisms for closed 3-manifold Y

$$\widecheck{HM}_{\bullet}(Y)\cong ECH(-Y)\cong HF^{+}(Y)$$

by Kutluhan-Lee-Taubes '12 or Colin-Ghiggini-Honda '12 and Taubes '10.

Motivation

Facts

- (Kronheimer-Mrowka '11) For a knot $K \subset S^3$, there is a spectral sequence from the reduced Khovanov homology $Khr(\bar{K})$ to $KHI(S^3, K)$, where $\bar{K}$ is the mirror of K.
- (Lim '09, Kronheimer-Mrowka '10) For a link $L \subset S^3$, we have $\chi_{\mathrm{gr}}(KHI(S^3, K)) = \Delta_L(t)$, where $\Delta_L(t)$ is the (single-variable) Alexander polynomial of L.
- (Scaduto '15) For a link $L \subset S^3$, there is a spectral sqeuence from the reduced odd Khovanov homology $Khr'(\bar{L})$ to $I^\sharp(\Sigma(L))$, where $\Sigma(L)$ is the double branched cover of S^3 over L.
- (Scaduto '15) For any closed 3-manifold Y, we have $\chi(I^\sharp(Y)) = |H_1(Y;\mathbb{Z})|$.

Motivation

Facts

The conjecture $SHI(M,\gamma) \cong SFH(M,\gamma) \otimes \mathbb{C}$ holds for

- S^3, lens spaces, $S^1 \times S^2$, and other simple manifolds by direct calculations;
- alternating links in S^3 (by spectral sequence and the alexander polynomial);
- double branched covers of S^3 over alternating links L (similar as above);
- some torus knots (Lobb-Zentner '13, Kronheimer-Mrowka '14, Hedden-Herald-Kirk '14, Daemi-Scaduto '19, *et al.*)
- some closed 3-manifolds obtained by surgeries on knots in S^3 (Lidman-Pinzón-Scaduto '20, Baldwin-Sivek '20, *et al.*).

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Main theorem

Theorem A (Baldwin-Li-Y. '20)

Suppose (M, γ) is a balanced sutured manifold and $\mathcal{H}$ is an (admissible) balanced diagram of (M, γ). Then we have

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leqslant \dim_{\mathbb{F}_2} SFC(\mathcal{H}).$$

Theorem B (Li-Y. '21)

Suppose (M, γ) is a balanced sutured manifold and $H = H_1(M; \mathbb{Z})/\mathrm{Tors}$. Then

$$\chi_{\mathrm{gr}}(SHI(M, \gamma)) = \chi_{\mathrm{gr}}(SFH(M, \gamma)) \in \mathbb{Z}[H]/\pm H.$$

Remark

- Li-Y. '20 proved Theorem A for $(1, 1)$ diagrams of knots in lens spaces.
- In preparation (Li-Y.): Theorem B can be proved for $H = H_1(M; \mathbb{Z})$.

Main theorem

For a balanced diagram $\mathcal{H}$, the generator of $SFC(\mathcal{H})$ can be calculated $\Longrightarrow$ Theorem A provides an upper bound on $\dim_{\mathbb{C}} SHI(M,\gamma)$.

For a balanced sutured manifold (M,γ), $\chi_{\mathrm{gr}}(SFH(M,\gamma))$ can be calculated by Fox calculus $\Longrightarrow$ Theorem B provides a lower bound on $\dim_{\mathbb{C}} SHI(M,\gamma)$.

When two bounds match, we have the following corollary.

Corollary

The conjecture $SHI(M,\gamma)\cong SFH(M,\gamma)\otimes\mathbb{C}$ holds for

- (1,1)-knots in lens spaces whose Alexander polynomials determine $\widehat{HFK}$, in particular, all torus knots, constrained knots (introduced by Y. 20), and $(-2,p,q)$-pretzel knots for $p,q\in 2\mathbb{N}+1$;
- strong L-space (introduced by Greene-Levine '16), for which there exists a balanced diagram $\mathcal{H}$ such that $\dim_{\mathbb{F}_2} SFC(\mathcal{H})=\dim_{\mathbb{F}_2} SFH(M,\gamma)$.

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Sketch of proofs: Theorem A

The proof of Theorem A is based on the following proposition.

Proposition A1 (Li-Y. '20 for one component, Baldwin-Li-Y. '20 for any case)

Suppose T is a properly embedded tangle in (M,γ) such that each component of T intersects each of $R_+(\gamma)$ and $R_-(\gamma)$ once. Suppose $M_T = M - \text{int}N(T)$ and $\gamma_T = \gamma \cup m_T$, where m_T is the union of meridians of each component of T.
If $[T] = 0 \in H_1(M, \partial M; \mathbb{Z})$, then

$$\dim_{\mathbb{C}} SHI(M,\gamma) \leqslant \dim_{\mathbb{C}} SHI(M_T,\gamma_T).$$

Remark

If $(M,\gamma) = (Y - \text{int}B^3, S^1)$, then $(M_T,\gamma_T) = (Y - \text{int}N(K), m \cup (-m))$ for some knot $K \subset Y$, where $[K] = 0 \in H_1(Y;\mathbb{Q})$. Then we have

$$\dim_{\mathbb{C}} I^{\sharp}(Y) \leqslant \dim_{\mathbb{C}} KHI(Y,K).$$

Sketch of proofs: Theorem A

For a given (admissible) balanced diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$ of (M, γ), we want to construct a tangle T satisfying the assumption of Proposition A1 and

$$\dim_{\mathbb{C}} SHI(M_T, \gamma_T) = \dim_{\mathbb{F}_2} SFC(\mathcal{H})$$

For simplicity, suppose $H_1(M, \partial M; \mathbb{Q}) = 0$, then we always have $[T] = 0$ and there is no admissible condition on $\mathcal{H}$.

Choose a point p_i in each component of $\Sigma \backslash \alpha \cup \beta$ that is disjoint from $\partial\Sigma$. Set

$$T = \bigcup_i [-1, 1] \times p_i \subset [-1, 1] \times \Sigma \subset M.$$

Sketch of proofs: Theorem A

Proposition A2 (Baldwin-Li-Y. '20)

Suppose $H_1(M, \partial M; \mathbb{Q}) = 0$. For above choice of T, we have

$$\dim_{\mathbb{C}} SHI(M_T, \gamma_T) = \dim_{\mathbb{F}_2} SFC(\mathcal{H})$$

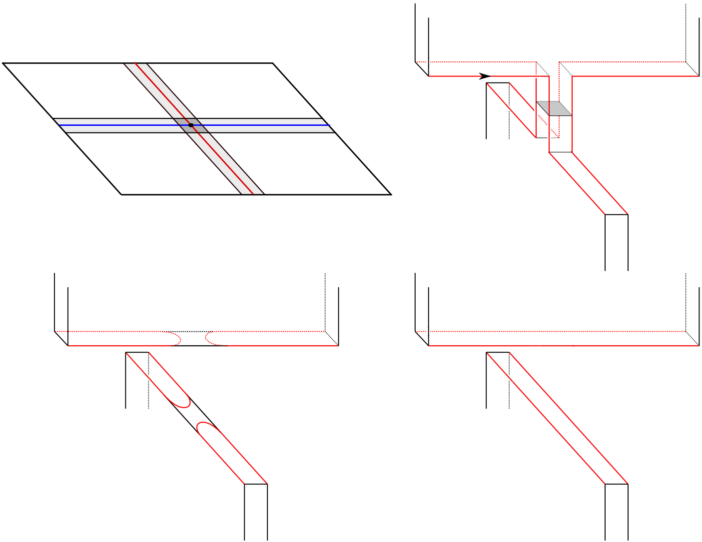
Proposition A3 (Baldwin-Li-Y. '20)

Suppose $D \subset M$ is a properly embedded disk with $|\partial D \cap \gamma| = 4$. Then

$$SHI(M, \gamma) \cong SHI(M', \gamma') \oplus SHI(M'', \gamma''),$$

where (M', γ') and (M'', γ'') are the decompositions of (M, γ) along D and $-D$, respectively.

Sketch of proofs: Theorem A



Sketch of proofs: Theorem B

The proof of Theorem B is based on the following proposition.

Proposition B1 (Lekili '13, Baldwin-Sivek '16)

Suppose (Y, R) is a closure of (M, γ). Define

$$SHF(M,\gamma) = HF(Y|R) = \bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(Y) \\ \langle c_1(\mathfrak{s}), R\rangle = 2g(R)-2}} HF^+(Y,\mathfrak{s}),$$

where HF^+ is the plus version of Heegaard Floer homology defined by Oszváth-Szabó '04. Then we have

$$SHF(M,\gamma) \cong SFH(M,\gamma).$$

Moreover, the isomorphism respects (nontorsion) spin^c structures and $\mathbb{Z}_2$-gradings.

Sketch of proofs: Theorem B

By Proposition B1, we only need to prove

$$\chi_{\mathrm{gr}}(SHI(M,\gamma)) = \chi_{\mathrm{gr}}(SHF(M,\gamma)),$$

where both spaces are defined by closures. In particular, we can choose the same closure (Y,R) and the same surfaces $S_1,\dots,S_n$ to induce the $\mathbb{Z}^n$-grading. Suppose

$$\cdots \to X \to Y \to Z \to X[1] \to \dots$$

is a long exact sequence, then there is a choice of signs such that

$$\chi(X) = \pm\chi(Y) \pm \chi(Z).$$

There are two long exact sequences we use, namely the surgery exact triangle and the bypass exact triangle.

Sketch of proofs: Theorem B

Proposition B2 (surgery exact triangle, Floer '90)

Suppose K is a knot in M. Let (M_i, γ_i) be obtained from (M, γ) by Dehn surgery along K with slope μ_i. If

$$\mu_1 \cdot \mu_2 = \mu_2 \cdot \mu_3 = \mu_3 \cdot \mu_1 = -1,$$

then there exists a long exact sequence

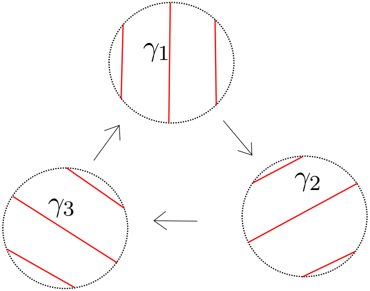
$$\cdots \to SHI(M_1, \gamma_1) \to SHI(M_2, \gamma_2) \to SHI(M_3, \gamma_3) \to SHI(M_1, \gamma_1)[1] \to \cdots$$

Sketch of proofs: Theorem B

Proposition B2 (bypass exact triangle, Baldwin-Sivek '18)

Suppose $\gamma_1, \gamma_2, \gamma_3$ are three sutures on M such that γ_i are the same except in a disk, where they look like as follows. Then there exists a long exact sequence

$$\cdots \to SHI(M,\gamma_1) \to SHI(M,\gamma_2) \to SHI(M,\gamma_3) \to SHI(M,\gamma_1)[1] \to$$



Sketch of proofs: Theorem B

Then we prove Theorem B by induction.

- The base case is $(M,\gamma)=(B^3,S^1)$, for which

$$\dim_{\mathbb{C}} SHI(M,\gamma)=\dim_{\mathbb{F}_2} SHF(M,\gamma)=1;$$

- In the case where M is a handlebody, we use the bypass exact triangle to make γ simper and then use the decomposition theorem (e.g. Proposition A3) to decrease the genus of the handlebody.
- In the general case, we use the surgery exact triangle to reduce (M,γ) to a sutured handlebody.

Thanks for your attention.